The Imagination Machine X: The Simplicial Structure of Compression and Extension

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Abstract

The *Imagination Machine* series develops a formal framework for embedded epistemic systems across nine papers, spanning epistemology, dynamical systems, predictive learning, institutional transmission, analogy, structural completion, ethics, theology, and categorical formulation. The present paper identifies the common formal structure underlying all of these constructions.

The compression and extension operations recurring throughout the series share four relational invariants with the face and degeneracy maps of simplicial sets. These invariants constitute an abstract mediating domain D_{abs} in the sense of *The Imagination Machine V* and *VI*: there exists a formal analogy between the series and the category of simplicial sets, and both are recoverable from D_{abs} by projection. Simplicial sets are the algebraically perfect instantiation of the four invariants. The series is the epistemically embedded instantiation, in which the fourth invariant—that compression after extension returns the original—holds at fixed points of the inference–implication dynamics rather than as a universal algebraic identity.

This framing retroactively illuminates the Koopman connection that appeared independently in two earlier papers. Linear evolution in observable space is a consequence of the first invariant—that compression preserves selected relational invariants while dropping indexical detail—shared by both instantiations of D_{abs}.

Contents

1 Introduction

The nine papers of the *Imagination Machine* series were not planned as a single formal structure. They developed sequentially, each paper extending or applying the framework established by its predecessors. By the time the ninth paper was complete, a retroactive question became available that could not have been asked earlier: what kind of mathematical object is the series itself?

The Imagination Machine IX answered part of that question by showing that the series forms a tower of functors between categories of structured spaces. Each paper corresponds to a layer in the tower, preserving selected relational invariants while discarding detail. The present paper asks whether those invariants have a known mathematical home.

They do. The compression and extension operations of the series share four structural properties with the face and degeneracy maps of simplicial sets. The series' own account of analogy, developed in *The Imagination Machine V* and formalized in *The Imagination Machine VI*, provides the right framework for stating this precisely: we construct a formal analogy $\mathcal{A} = (X, Y, M, P)$ between the series and the category of simplicial sets, identify the preserved relations P, and exhibit the abstract mediating domain D_{abs} of which both are instances.

One of the four preserved relations requires explicit qualification. The mixed simplicial identity $d_i s_j = \text{id}$ says that compression after extension returns the original as an algebraic equation holding universally. The analogous condition in the series is $T(w^*) = w^*$: compression after extension returns the original, but only at a fixed point of the inference–implication dynamics, and only after convergence. This asymmetry is noted in Section 5. It locates precisely where the two instantiations of D_{abs} differ, and that location is the epistemically interesting territory: the series describes what happens in the approach to the simplicial limit, while simplicial sets describe the limit itself.

The Koopman connection, addressed in Section 7, follows from the first preserved relation rather than requiring separate derivation.

2 Simplicial Sets: The Relevant Structure

We recall the relevant definitions.

Definition 1 (Simplicial Set)**.** *A simplicial set X consists of sets X_n of n-simplices for each $n \geq 0$, together with:*

- ***Face maps*** $d_i : X_n \to X_{n-1}$ *for* $0 \leq i \leq n$*, and*

- ***Degeneracy maps*** *$s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n$,*

satisfying the simplicial identities:

$$d_i d_j = d_{j-1} d_i \quad \text{if } i < j \tag{1}$$

$$s_i s_j = s_{j+1} s_i \quad \text{if } i \leq j \tag{2}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases} \tag{3}$$

An n-simplex is a coherent relational configuration among $n+1$ objects. A face map d_i drops the i-th object, producing a lower-dimensional face. A degeneracy map s_i repeats the i-th object, producing a higher-dimensional simplex containing the original as a degenerate case. The simplicial identities are the conditions under which dropping and extending cohere regardless of order.

Remark 1. *The simplicial identities (1)–(3) are algebraic equations between morphisms. The present paper argues that the series shares the structural pattern these identities express. What is preserved across the analogy is the pattern, not the equations themselves.*

3 Analogy as Mediating Structure

We recall the formal account of analogy from *The Imagination Machine V* and *VI*.

Definition 2 (Analogy, from TIM V)**.** *An analogy between a source domain $D_s = (O_s, A_s, R_s, S_s, T_s)$ and a target domain $D_t = (O_t, A_t, R_t, S_t, T_t)$ is a tuple $\mathcal{A} = (X, Y, M, P)$ where $X \subset O_s$, $Y \subset O_t$, $M : X \to Y$ is a mapping of objects, and $P \subset R_s \cap R_t$ is a set of relations preserved by M.*

Definition 3 (Abstract Mediating Domain, from TIM V)**.** *Given an analogy $\mathcal{A} = (X, Y, M, P)$, the abstract mediating domain $D_{\text{abs}} = (O_{\text{abs}}, A_{\text{abs}}, R_{\text{abs}}, S_{\text{abs}}, T_{\text{abs}})$ has objects $O_{\text{abs}} = \{(x, M(x)) \mid x \in X\}$, abstract relations $R_{\text{abs}} = P$, and belief set T_{abs} containing $r((x_1, M(x_1)), \ldots, (x_k, M(x_k)))$ whenever $r(x_1, \ldots, x_k) \in T_s$ for $r \in P$. The canonical projections $\pi_s(x, M(x)) = x$ and $\pi_t(x, M(x)) = M(x)$ exhibit D_s and D_t as instantiations of D_{abs}.*

The present paper constructs an analogy in this sense between the *Imagination Machine* series and the category of simplicial sets. The source domain D_s is the series, whose objects of interest are the compression and extension operations recurring across all nine papers.

The target domain D_t is the category of simplicial sets, whose objects include face maps, degeneracy maps, the simplicial identities, horns, and the Kan condition. The preserved relations P are the four structural invariants identified in Section 4.

3.1 The Source Domain: Operations of the Series

The objects $X \subset O_s$ are the recurring operations of the series, grouped by structural role.

Compression operations X_C: the inference map $F : \Gamma \to W$ of *The Imagination Machine I*; the two-stage institutional compression of *The Imagination Machine IV*; the construction of the abstract mediating domain from source and target domains in *The Imagination Machine V* and *VI*; the moral universalization operator of *The Imagination Machine VII*; the geometric projection $\pi : \mathcal{B} \to \mathcal{E}$ of *The Imagination Machine VIII*; the graph quotient operation of *The Imagination Machine XI*.

Extension operations X_E: the implication map $g : W \to \Gamma$ of *The Imagination Machine I*; generative inheritance of *The Imagination Machine IV*; analogical reasoning steps and horn filling of *The Imagination Machine V* and *VI*; the embedding map $\iota : \mathcal{E} \hookrightarrow \mathcal{B}$ of *The Imagination Machine VIII*; graph completion of *The Imagination Machine XI*.

3.2 The Target Domain: Simplicial Structure

The objects $Y \subset O_t$ are the canonical simplicial operations: face maps d_i, degeneracy maps s_i, the simplicial identities (1)–(3), horns Λ^n_k, and the Kan horn-filling condition.

3.3 The Mapping

The mapping $M : X \to Y$ sends compression operations to face maps and extension operations to degeneracy maps:

$$M(x) = \begin{cases} d_i & \text{if } x \in X_C \\ s_i & \text{if } x \in X_E. \end{cases}$$

The index i is not fixed by M; the mapping identifies structural role rather than position in a particular simplex.

4 The Four Preserved Relations

The preserved relations P are the structural invariants shared by both domains.

P1. Compression reduces representational complexity while preserving selected relational invariants. In the series: F drops indexical detail while preserving the relational structure of observations (*TIM I*); institutional summarization drops redundancy while preserving proposed revisions (*TIM IV*); analogical abstraction drops object-level attributes while preserving relational predicates $P \subset R_s \cap R_t$ (*TIM V*); moral universalization drops agent-specific content while preserving the action–motivation structure (*TIM VII*); geometric projection drops one dimension while preserving the relational structure of $\mathcal{B}$ at reduced dimension (*TIM VIII*). In simplicial sets: d_i drops the i-th vertex while preserving the relational structure of the remaining vertices.

P2. Extension reconstructs richer structure consistent with preserved invariants. In the series: g generates a full observational profile from a compressed world model (*TIM I*); generative inheritance reconstructs the closure mechanism from a transmitted fixed point (*TIM IV*); horn filling completes a partial simplicial configuration (*TIM VI*); ι embeds the three-dimensional cross-section into the four-dimensional containing structure (*TIM VIII*); graph completion infers missing relational structure (*TIM XI*). In simplicial sets: s_i extends an n-simplex to an $(n+1)$-simplex by repeating the i-th vertex, producing a higher-dimensional structure consistent with the original.

P3. The output type of extension matches the input type of compression. In the series: g produces observational profiles of the type that F consumes; filled simplices in *The Imagination Machine VI* are simplices eligible for further horn configurations; abstract mediating domains are domains eligible for further analogies (Proposition 2 of *TIM VI*). In simplicial sets: $s_i(x) \in X_{n+1}$ is a simplex and therefore a valid input to face maps at dimension $n+1$.

P4. Compression after extension at stability returns the original. This relation holds with a qualification addressed in Section 5.

5 The Fixed-Point Qualification

In simplicial sets, the mixed identity (3) includes $d_i s_j = \mathrm{id}$ when $i = j$ or $i = j+1$: compression after extension returns the original as an algebraic identity holding universally for every simplex.

In the series, the analogous condition is $T(w^*) = w^*$, where $T = F \circ g$. Compression after extension returns the original—but only at a fixed point $w^* \in W^*$, after the inference–implication loop has converged. At intermediate steps $T(w) \neq w$ in general. The same structure appears in the reinforcement learning closure of *The Imagination Machine III*, the universalization fixed point of *The Imagination Machine VII*, and the self-consistency of the cross-section with the containing structure in *The Imagination Machine VIII*: in each case the condition holds at the fixed point of a convergent dynamical process.

P4 therefore holds in the series in the following qualified form: compression after extension at the stable point of the compression–extension dynamics returns the original. Simplicial sets instantiate this with trivial dynamics—every simplex is already at its stable point. The series instantiates this with nontrivial dynamics—stability is achieved asymptotically under the pressure of observation and inference.

Remark 2. *This asymmetry locates precisely where the two instantiations of D_{abs} differ. Simplicial sets are the limit case in which every horn fills immediately and the mixed identity holds everywhere. The series describes the dynamics of approach to that limit from within an embedded epistemic position. The abstract mediating domain contains both, related by the difference between algebraic universality and asymptotic convergence.*

6 The Abstract Mediating Domain

Proposition 1 (Formal Analogy Between the Series and Simplicial Sets)**.** *There exists a formal analogy $\mathcal{A} = (X, Y, M, P)$ between the Imagination Machine series D_s and the category of simplicial sets D_t, with abstract mediating domain D_{abs} characterized by the four relations $P = \{P1, P2, P3, P4^*\}$, where $P4^*$ is the qualified form of P4 stated in Section 5. Both D_s and D_t are instantiations of D_{abs}, recoverable by the projections π_s and π_t.*

Proof. We verify that each relation in P is instantiated in both D_s and D_t.

P1 holds in D_s by the results cited in Section 4 for each element of X_C. P1 holds in D_t by definition of face maps.

P2 holds in D_s by the results cited in Section 4 for each element of X_E. P2 holds in D_t by definition of degeneracy maps.

P3 holds in D_s by Proposition 2 of *The Imagination Machine VI*, which establishes that in each framework of the series the extension operation produces structures of the same type as the inputs to the compression operation. P3 holds in D_t since $s_i(x) \in X_{n+1}$ is a simplex eligible as input to $d_j : X_{n+1} \to X_n$.

$P4^*$ holds in D_s by the fixed-point results of *The Imagination Machine I* ($T(w^*) = w^*$), *III* (the reinforcement learning closure (w^*, π^*)), *VII* (the universalization fixed point), and *VIII* (the self-consistency of $\mathcal{E}$ within $\mathcal{B}$). $P4^*$ holds in D_t by the degenerate cases of the mixed identity (3).

Since all four relations in P are instantiated in both domains, the analogy $\mathcal{A}$ is well-defined and D_{abs} is the abstract mediating domain of which both are instances. □

Remark 3 (Self-Demonstration)**.** *The construction of Proposition 1 is itself an instance of analogical abstraction: two domains are identified, a mapping between their operations is* *exhibited, preserved relations are stated, and an abstract mediating domain is constructed. This is the operation that The Imagination Machine V defines and The Imagination Machine VI identifies as an instantiation of the extension schema. The construction that establishes the correspondence is an instance of the correspondence it establishes.*

7 The Koopman Connection

The Koopman representation appears twice in the series. In *The Imagination Machine III*, the relational observables $z_{ij}(t) = e^{i\Delta_{ij}(t)}$ of a quasi-periodic dynamical system evolve linearly in observable space even though the underlying state dynamics are nonlinear. In *The Imagination Machine IX*, this is formalized as a functor $\mathcal{O} : \mathbf{Struct} \to \mathbf{Obs}$ mapping conceptual structures to spaces of observables in which dynamics become tractable.

Both appearances present Koopman linearity as a feature of the particular observables chosen. The present paper observes that it is a consequence of P1.

Proposition 2 (Koopman Linearity as Consequence of P1)**.** *Let $\varphi \in X_C$ be any compression operation satisfying P1, and let states evolve according to a rule F. Then the induced evolution on the image of φ is linear in the space of relational invariants preserved by φ.*

Proof. By P1, φ retains exactly the relational invariants in its image and drops all indexical content not captured by those invariants. Two states x, x' are identified by φ if and only if they agree on all preserved invariants. The induced evolution on the quotient $X/\ker(\varphi)$ is therefore determined solely by the action of F on those invariants, independently of the dropped indexical content. This is the Koopman representation for φ: nonlinear dynamics on state space become linear on the space of preserved relational invariants. □

Remark 4. *The relational phase observables $(\cos\Delta_{ij}, \sin\Delta_{ij})$ of The Imagination Machine III are the relational invariants preserved by the compression that drops absolute phases. Their linear evolution is the instance of Proposition 2 for that specific compression. Since P1 holds for every element of X_C, the same linearity holds for every compression operation in the series and, by the analogy $\mathcal{A}$, for every face map in the target domain.*

8 The Series as a Whole

8.1 What the Abstract Mediating Domain Reveals

The construction of D_{abs} reveals three things not visible from within any individual paper.

First, the coherence of the series is structural. The papers share four relational invariants constituting a genuine abstract domain with a known mathematical instantiation in simplicial sets.

Second, the Koopman linearity of *The Imagination Machine III* and *IX* is a consequence of P1 rather than an independent result. Any compression operation satisfying P1 induces Koopman-linear dynamics on its image.

Third, the extension schema of *The Imagination Machine VI* is itself an element of X_E, mapped by M to the simplicial extension operation. The series contains, as one of its operations, the construction that produced D_{abs}.

8.2 The Kan Condition

The Kan condition on a simplicial set requires that every horn $\Lambda^n_k \to X$ admits a filler $\Delta^n \to X$: no partial relational configuration goes unextended. This is the perfect instantiation of P2 and P3 combined.

The series instantiates the Kan condition in the sense of $P4^*$: every partial relational configuration within the framework admits a coherent completion at the fixed point of the relevant dynamics. This is established by Theorem 1 of *The Imagination Machine VI* for holonic composition, simplicial horn filling, and analogical abstraction; by the fixed-point results of *The Imagination Machine I* and *VII* for the epistemic and moral domains; and by the embedding structure of *The Imagination Machine VIII* for the geometric domain. The series is therefore an embedded instantiation of the structure that Kan complexes instantiate perfectly.

8.3 The Theological Register

The Imagination Machine VIII observed that the geometric theology underlying the series and the formal framework of the series are two cross-sections of the same structure. The present paper adds precision: both are instantiations of D_{abs}, related by the analogy $\mathcal{A}$ in the same way that the series and simplicial sets are related.

The medieval formula—God is a circle whose center is everywhere and whose circumference is nowhere—describes the containing structure of a Kan complex as encountered from within one of its faces: an interior nowhere locatable from within the faces and yet participating in every face. The embedded observer's epistemic situation instantiates the same abstract structure from the inside, approaching the fixed point rather than occupying it.

9 Conclusion

The *Imagination Machine* series and the category of simplicial sets share an abstract mediating domain D_{abs} characterized by four relational invariants: compression preserves selected invariants while reducing complexity (P1); extension reconstructs richer structure consistent with preserved invariants (P2); the output type of extension matches the input type of compression (P3); and compression after extension at the stable point of the dynamics returns the original (P4*, qualified). Simplicial sets are the algebraically perfect instantiation of these four conditions. The series is the epistemically embedded instantiation, in which P4 holds asymptotically rather than universally.

The Koopman linearity that appeared independently in two earlier papers is a consequence of P1 shared by both instantiations. The extension schema of *The Imagination Machine VI* is itself an element of the series mapped by $\mathcal{A}$ to the simplicial extension operation. The construction of this paper instantiates the analogical abstraction it formalizes.

The series propagates itself forward by being what it is.

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